Hawking Radiation from Squashed Kaluza-Klein Black Holes — A Window to Extra Dimensions —

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We explore the obsevability of extra dimensions through 5-dimensional squashed Kaluza-Klein black holes residing in the Kaluza-Klein spacetime. With the expectation that the Hawking radiation reflects the 5-dimensional nature of the squashed horizon, we study the Hawking radiation of a scalar field in the squashed black hole background. As a result, we show that the luminosity of Hawking radiation tells us the size of the extra dimension, namely, the squashed Kaluza-Klein black holes open a window to extra dimensions.

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Introduction — It is believed that the superstring theory is a promising candidate for the unified theory of everything. As the superstring theory can be consistently formulated only in 10-dimensions, the existence of extra dimensions should be regarded as the prediction of the theory. Therefore, it is of interest to see if these extra dimensions are observable or not. Here, we will seek the possibility to detect extra dimensions through black holes.

As is well known, black holes are not black in the semiclassical theory. Rather, they emit the Hawking radiation [1]. For a 4-dimensional Schwarzschild black hole, the luminosity of Hawking radiation can be deduced as $L = 1/1920\pi r_+^2$, where we have calculated only the s-wave contribution which is dominant at low energy. In this conventional case, the luminosity of the Hawking radiation is determined by the horizon size r_+.

In the higher dimensional Kaluza-Klein spacetime, it is natural to regard black holes as the direct product of the conventional 4-dimensional black hole spacetime and the internal compact space. In the 5-dimensional case, it can be described by the black string. In this case, it is difficult to observe the extra dimension as long as the size of the compact space is sufficiently small. This is because, at low energy, the effects of the extra dimension is merely absorbed into the definition of r_+. In order to see the extra dimension, we need to see the effect where Kaluza-Klein modes play a role.

However, squashed Kaluza-Klein black holes recently investigated in [3] (see also [4, 5]) have changed the situation. The squashed Kaluza-Klein black holes look like the 5-dimensional black holes [6] in the vicinity of the horizon. Intriguingly, the spacetime far from the black holes is locally the direct product of the 4-dimensional Minkowski spacetime and the circle. In this sense, the 5-dimensional squashed Kaluza-Klein black holes reside in Kaluza-Klein spacetime. Since the Hawking radiation is considered as the phenomena associated with the event horizon, it is reasonable to expect that we can observe the extra dimension by looking at the Hawking radiation. The purpose of this paper is to show this is indeed the case.

Kaluza-Klein Black Holes with Squashed Horizons— Let us review the Kaluza-Klein black holes with squashed horizons [3]. We consider the 5-dimensional gravity coupled with a $U(1)$ gauge field. The action is given by

$$S = \frac{1}{16\pi G}\int d^5x\sqrt{-g}\left[R - F^{\mu\nu}F_{\mu\nu}\right], \tag{1}$$

where G is the 5-dimensional Newton constant, $g_{\mu\nu}$ is the metric, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of the 5-dimensional $U(1)$ gauge field A_μ. We have black hole solutions with the metric given by

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{k^2(r)}{f(r)}dr^2 \\ &\quad + \frac{r^2}{4}k(r)d\Omega^2 + \frac{r^2}{4}\left(d\psi + \cos\theta d\phi\right)^2, \end{aligned} \tag{2}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric of the unit sphere and

$$f(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^4}, k(r) = \frac{(r_\infty^2 - r_+^2)(r_\infty^2 - r_-^2)}{(r_\infty^2 - r^2)^2}. \tag{3}$$

Here, the coordinate ranges are $0 \le \theta < 2\pi, 0 \le \phi < 2\pi, 0 \le \psi < 4\pi$. The gauge potential is given by

$$A = \pm\frac{\sqrt{3}}{2}\left(\frac{r_+r_-}{r^2} - \frac{r_+r_-}{r_\infty^2}\right)dt. \tag{4}$$

It is easy to see the coordinate singularity at r_+ corresponds to the outer horizon of the black hole. The inner horizon r_- is analogous to that of Reissner-Nordström black holes. The position r_∞ corresponds to the spatial infinity. It should be noted that the shape of the horizon is a deformed sphere determined by the parameter $k(r_+)$. The solution reveals 5-dimensional nature in the vicinity of the horizon $r_\pm < r \ll r_\infty$.

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On the other hand, the geometry becomes the 4-dimensions with a small circle at infinity. To see this, we use the coordinate transformation $\rho = \rho_0 r^2/(r_\infty^2 - r^2)$. It yields the relation $r_\infty^2 = 4\left(\rho_+ + \rho_0\right)\left(\rho_- + \rho_0\right)$, where $\rho_\pm = \rho_0 r_\pm^2/(r_\infty^2 - r_\pm^2)$. Under this coordinate transformation, the metric reduces to

$$\begin{aligned} ds^2 &= -F(\rho)d\tau^2 + \frac{K^2(\rho)}{F(\rho)}d\rho^2 + \rho^2 K^2(\rho)d\Omega^2 \\ &\quad + \frac{r_\infty^2}{4K^2(\rho)}\left(d\psi + \cos\theta d\phi\right)^2, \end{aligned} \tag{5}$$

where

$$F = \left(1 - \frac{\rho_+}{\rho}\right)\left(1 - \frac{\rho_-}{\rho}\right), \quad K^2 = 1 + \frac{\rho_0}{\rho}. \tag{6}$$

We have defined the proper time $\tau = 2\rho_0 t/r_\infty$ for the observer at infinity. We note that even when ρ_0 is negative, the metric (5) describes a black hole geometry if the parameters satisfy $-\rho_0 = |\rho_0| < \rho_\pm$[7]. In the region $\rho \gg \rho_0$, the geometry locally looks like the black string. In the case $\rho_+ < \rho_0$, the black holes look like 5-dimensional in the region $\rho < \rho_0$. The twist of the geometry is essential for our consideration.

Given the metric, we can calculate various physical quantities. The surface gravity is given by

$$\kappa_+ = \frac{\rho_+ - \rho_-}{2\rho_+^2}\sqrt{\frac{\rho_+}{\rho_+ + \rho_0}}. \tag{7}$$

Using the timelike Killing vector $\xi = \partial/\partial\tau$ normalized at the spatial infinity, we can define the mass of the black hole as

$$M = -\frac{3}{32\pi G}\int_\infty dS_{\mu\nu}\nabla^\mu\xi^\nu = \frac{3\pi r_\infty}{4G}\left(\rho_+ + \rho_-\right), \tag{8}$$

where the integral is taken over the three dimensional sphere at the spatial infinity. The charge of the black hole is also calculated as

$$Q = \frac{1}{8\pi G}\int_\infty dS_{\mu\nu}F^{\mu\nu} = \frac{\sqrt{3}\pi}{G}r_\infty\sqrt{\rho_+\rho_-}. \tag{9}$$

We should notice that the Newton constant for the remote observer should be $G_4 = G/2\pi r_\infty$. Therefore, we cannot see the extra dimension from the observation of the mass M and the charge Q, because the size of the extra dimension is absorbed into the definition of G_4. We need another observable to probe the extra dimension.

Mode Functions— Let us consider the field equation of the massless scalar field Φ in the spacetime (5), $\Box\Phi = 0$. Fortunately, this turns out to be a separable equation. Taking a separable ansatz $\Phi = e^{-i\omega\tau}R(\rho)e^{-i\lambda\psi}e^{im\phi}S(\theta)$, we obtain the equation

$$\begin{aligned} &\frac{F}{\rho^2K^2}\frac{\partial}{\partial\rho}\left[\rho^2 F\frac{\partial R}{\partial\rho}\right] + \left[\omega^2 - \frac{4\lambda^2K^2F}{r_\infty^2}\right]R \\ &\quad = \frac{\left\{\ell(\ell+1) - \lambda^2\right\}F}{\rho^2K^2}R \end{aligned} \tag{10}$$

for the radial part. The angular part becomes

$$\Delta S - \frac{\lambda^2\cos^2\theta}{\sin^2\theta}S - \frac{2\lambda m\cos\theta}{\sin^2\theta}S = -\left\{\ell(\ell+1) - \lambda^2\right\}S, \tag{11}$$

where Δ is the Laplacian for the unit sphere. The solutions for Eq.(11) are called the spin-weighted spherical functions. The spectrum is determined from the regularity. We also have conditions $\ell \geq \lambda$ and $m = -\ell, -\ell+1, \cdots, \ell$. The periodicity requires 2λ, $2m$, and $\lambda \pm m$ are integers.

To understand the qualitative behavior of the radial mode functions, it is convenient to use new variables $d\rho_* = Kd\rho/F$, $R(\rho) = Y(\rho)/\sqrt{\rho^2K}$. The radial equations can be transformed to the Schrödinger form

$$-\frac{d^2Y}{d\rho_*^2} + VY = \omega^2 Y, \tag{12}$$

where

$$\begin{aligned} V &= \frac{4\lambda^2}{r_\infty^2}K^2F + \frac{(\ell(\ell+1) - \lambda^2)}{\rho^2K^2}F + \frac{FF'}{\rho K^2} + \frac{3}{4}\frac{F^2}{\rho^2K^2} \\ &\quad + \frac{1}{2}\frac{F^2K''}{K^3} - \frac{3}{4}\frac{F^2K'^2}{K^4} + \frac{1}{2}\frac{FF'K'}{K^3}. \end{aligned} \tag{13}$$

From this formula, we can read off the effective potential

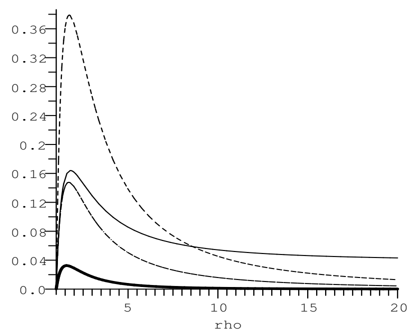

FIG. 1: The effective potentials $V(\rho)$ for the radial modes are depicted. We set $\rho_- = 0.2$, $\rho_+ = 1$ and $\rho_0 = 2$. The thick line represents $\ell = \lambda = 0$. The dash-dot and dash lines correspond to $\ell = 1, \lambda = 0$ and $\ell = 2, \lambda = 0$, respectively. The thin line represents $\ell = 1, \lambda = 0.5$.

(see Fig.1). For higher eigenvalues λ, ℓ, the height of the potential becomes higher. At low energy, hence, only the modes $\lambda = \ell = 0$ are relevant.

Hawking Radiation with Greybody Factors— In the semi-classical picture, the black hole can radiate particles. In fact, the black hole can be regarded as a black-body with the Hawking temperature $T_{BH} = \kappa_+/2\pi$. The greybody factors modify the spectrum of emitted particles from that of a perfect blackbody. The luminosity of the Hawking radiation with greybody factors is then given by the formula

$$L = \int_0^\infty \frac{d\omega}{2\pi}|\mathcal{A}(\omega)|^2\frac{\omega}{\exp\left(\omega/T_{\rm BH}\right) - 1}. \tag{14}$$

The boundary condition to calculate the greybody factor should be

$$R(\rho) \sim \begin{cases} \tilde{t}(\omega) e^{i\omega\rho_*}, & \rho_* \to \infty \\ e^{i\omega\rho_*} + \tilde{r}(\omega) e^{-i\omega\rho_*}, & \rho_* \to -\infty \end{cases}, \qquad (15)$$

where $\tilde{t}(\omega)$ and $\tilde{r}(\omega)$ are the transmission and reflection coefficients, respectively. However, the boundary condition for the conventional calculation is different. The absorption probability $|\mathcal{A}|^2$ is computed from solutions to the wave equation which have unit incoming flux from infinity and no outgoing flux from the past horizon:

$$R(\rho) \sim \begin{cases} e^{-i\omega\rho_*} + \mathcal{R}(\omega) e^{i\omega\rho_*}, & \rho_* \to \infty \\ \mathcal{T}(\omega) e^{-i\omega\rho_*}, & \rho_* \to -\infty \end{cases}, \qquad (16)$$

where $\mathcal{T}$ and $\mathcal{R}$ are, respectively, the transmission and reflection coefficients for this boundary condition. The well known relation $\tilde{t}(\omega) = \mathcal{T}(\omega)$ guarantees the equivalence of both boundary conditions. Here, we just follow this conventional procedure. The absorption probability is the difference of the incoming and out going flux at infinity. Thus, the absorption probability $|\mathcal{A}|^2$ can be calculated from the reflection coefficient $\mathcal{R}(\omega)$ as $|\mathcal{A}|^2 = |\tilde{t}(\omega)|^2 = |\mathcal{T}(\omega)|^2 = 1 - |\mathcal{R}(\omega)|^2$.

The wave equation we should solve is

$$\frac{F}{\rho^2}\frac{d}{d\rho}\left[\rho^2 F \frac{dR}{d\rho}\right] + \omega^2\left(1 + \frac{\rho_0}{\rho}\right) R = 0. \qquad (17)$$

Here, we put $\lambda = \ell = 0$ since we will consider the low energy regime. We can not solve the above equation analytically, then we have to resort to an asymptotic expansion technique. First, we separate the region for the coordinate ρ into three parts: the near horizon $\rho_+ \lesssim \rho$, the intermediate region $\rho \sim \rho_I$, and the far region $\rho \gg \rho_\pm, \rho_0$. Here, ρ_I is some reference point where $\omega\rho_I \ll 1$ is satisfied. The result does not depend on the specific choice of ρ_I as we will see soon. We can solve the equation in each region where certain small terms are neglected. By matching these solutions in the overlapping regions, we obtain an approximate solution for the wave equation. Hence, we can determine the absorption probability.

In the near horizon $\rho_+ \lesssim \rho$, we have

$$\rho^2 F \frac{d}{d\rho}\left[\rho^2 F \frac{dR}{d\rho}\right] + \omega^2 \rho_+^4 \left(1 + \frac{\rho_0}{\rho_+}\right) R = 0. \qquad (18)$$

Here, we have approximately put $\rho \sim \rho_+$ in the second term. This can be solved by using the transformation $dy/d\rho = 1/\rho^2 F$. Notice that $y \sim (1/\rho_+)\log(\rho - \rho_+) \to -\infty$ at the horizon. As we are considering the ingoing waves from the infinity, we do not have waves coming out from the horizon, namely, we only have ingoing waves at the horizon. Taking into account this ingoing boundary condition, we obtain the solution as

$$R_{NH} = a_- \exp\left(-i\omega\rho_+^2\sqrt{1 + \frac{\rho_0}{\rho_+}}\ y\right). \qquad (19)$$

In the regime $\rho > \rho_+$, the solution (19) becomes

$$R_{NH} = a_- \left[1 - i\omega\rho_+\sqrt{1 + \frac{\rho_0}{\rho_+}}\log(\rho - \rho_+)\right]. \qquad (20)$$

In the intermediate zone $\rho \sim \rho_I$, Eq.(17) reduces to

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left[\rho^2 F \frac{dR}{d\rho}\right] = 0. \qquad (21)$$

Here, we have assumed the $\omega\rho_I \ll 1$. The solution is given by

$$R_I = b_+ + b_- \int \frac{d\rho}{\rho^2 F}. \qquad (22)$$

In the inner region $\rho \sim \rho_+$, we obtain

$$R_I = b_+ + \frac{b_-}{\rho_+}\log(\rho - \rho_+), \qquad (23)$$

where we approximately put $\rho^2 F = \rho_+ \rho F$. This can be matched with the near horizon solution (20). In the outer region $\rho > \rho_I > \rho_+$, we can set $F \sim 1$. Hence, we have

$$R_I = b_+ - b_- \frac{1}{\rho}. \qquad (24)$$

In the far zone $\rho \gg \rho_\pm, \rho_0$, Eq.(17) becomes

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left[\rho^2 \frac{dR}{d\rho}\right] + \omega^2 R = 0. \qquad (25)$$

This can be solved analytically as

$$R_F = \frac{c_+}{\sqrt{\rho}} J_{1/2}(\omega\rho) + \frac{c_-}{\sqrt{\rho}} N_{1/2}(\omega\rho). \qquad (26)$$

In the region $\omega\rho \ll 1$, we get

$$R_F \sim \frac{2c_+}{\sqrt{\pi}}\left(\frac{\omega}{2}\right)^{1/2} - \frac{c_-}{\sqrt{\pi}\rho}\left(\frac{\omega}{2}\right)^{-1/2}. \qquad (27)$$

This can be matched with the intermediate solution (24). There exists the matching region $\rho_I < \rho < \omega^{-1}$ for low energy modes $\omega\rho_I \ll 1$.

From solutions (20) and (23), we can read off the matching conditions

$$a_- = b_+, \quad -i\omega\rho_+\sqrt{1 + \frac{\rho_0}{\rho_+}}a_- = \frac{b_-}{\rho_+}. \qquad (28)$$

and, from solutions (24) and (27), we obtain

$$b_+ = 2c_+\left(\frac{\omega}{2}\right)^{1/2}, \quad -b_- = -c_-\left(\frac{\omega}{2}\right)^{-1/2}. \qquad (29)$$

In the asymptotic region $\omega\rho \gg 1$, the solution in the far region can be expanded as

$$\begin{aligned} R_F &= \frac{-ic_+ - c_-}{2\rho}\frac{1}{\sqrt{2\pi\omega}}\exp(i\omega\rho) \\ &\quad + \frac{ic_+ - c_-}{2\rho}\frac{1}{\sqrt{2\pi\omega}}\exp(-i\omega\rho) \end{aligned} \qquad (30)$$

Therefore, the reflection coefficient is given by $\mathcal{R} = (ic_- - c_+)/(ic_- + c_+)$. Thus, we obtain the absorption probability

$$|\mathcal{A}|^2 = 4\left(\omega\rho_+\right)^2\sqrt{1+\frac{\rho_0}{\rho_+}}. \tag{31}$$

This result can be understood from the behavior of the effective potential (Fig.2). The larger ρ_0 yields the lower peak of the potential. Hence, more radiations can be transmitted to the infinity.

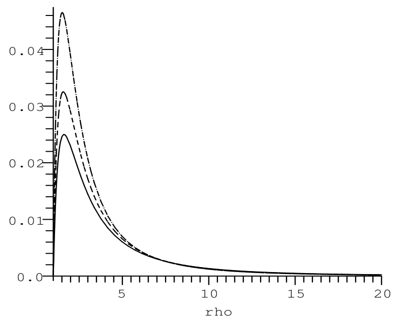

FIG. 2: The ρ_0 dependence of the effective potential $V(\rho)$ is plotted. Here, we set $\rho_- = 0.2$, $\rho_+ = 1$ and $\lambda = \ell = 0$. The dash-dot, dash, and line correspond to $\rho_0 = 1, 2, 3$ respectively.

Discussion— We have considered a scalar field in Kaluza-Klein black hole with the squashed horizon and obtained the greybody factor (31). Now, combining Eqs. (7), (14), and (31), we have the luminosity of the Hawking radiation

$$L = \frac{1}{1920\pi\rho_+^2}\left(1-\frac{\rho_-}{\rho_+}\right)^4\left(1+\frac{\rho_0}{\rho_+}\right)^{-3/2}. \tag{32}$$

In the the limit, $\rho_0 \to 0$, we obtain the metric (5) with $K^2 = 1$ which locally has the geometry of the black string. In this limit, Eq.(32) reduces to the standard formula for the 4-dimensional charged black holes. Therefore, without squashed horizons, we never notice the extra dimension even if we can detect Hawking radiations. For black holes with $\rho_0 \gg \rho_+$, the luminosity of Hawking radiation is significantly suppressed compared with that of the conventional 4-dimensional black holes. For the negative ρ_0, it is highly enhanced. This ρ_0 dependence allows us to see the extra dimension.

Now we come to our conclusion. Three observable quantities M, Q and L determine $\rho_\pm$ and ρ_0. Therefore, we can calculate the size of the extra dimension r_∞. Thus, in principle, we can observe the extra dimension through the Hawking radiation. Namely, the squashed black holes could be a window to extra dimensions.

Although we have discussed a 5-dimensional example, squashed higher dimensional black holes residing in higher dimensional Kaluza-Klein spacetime might exist. Attempts to find explicit solutions are important. It is of interest to extend our analysis to more general situations [8, 9, 10, 11, 12, 13]. We also need to study the stability of the squashed Kaluza-Klein black holes. There may exists the Gregory-Laflamme type instability [14]. Moreover, we can consider the squashed black holes in the braneworld context. Although inhomogeneities in the perpendicular direction to the brane and the radion make the situation complicated [15], the possibility of twists must provide us rich physical phenomena.

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